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Monte Carlo Simulation

SYS 611: Systems Modeling and Simulation

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Agenda

1. Monte Carlo Simulation
2. Buffon's Needle Activity

Reference: S.M. Ross, "Statistical Analysis of Simulated Data," Ch. 8 in *Simulation*, 2012.

J.V. Farr, "Review of Probability and Statistics," Ch. 3 in *Simulation of Complex Systems and Enterprises*, Stevens Institute of Technology, 2007.



Monte Carlo Simulation





Monte Carlo Simulation

Monte Carlo simulation solves a problem (possibly deterministic in nature) by statistically analyzing samples from a stochastic model

- Developed in 1940s classified research by von Neumann, Ulam, Fermi, Metropolis (& others)
- Code named by Ulam and Metropolis in reference to *Monte Carlo* casino in Monaco
- Early applications limited due to computation (ENIAC: 1st general-purpose computer in 1946)



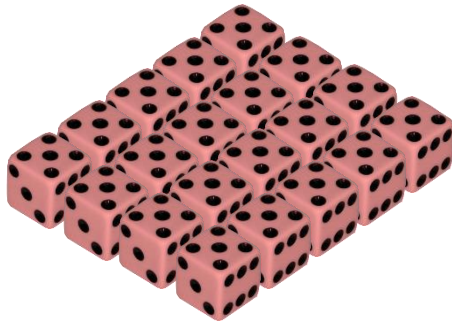
Monte Carlo Approach

1. Identify **elementary state variables** and **random (state) variables** with **probability distributions**
2. Identify **derived (state) variables** and their functional form
3. Determine **number of samples** required or other convergence criteria
4. For each sample, **generate RVs** and compose and **record derived (state) variables**
5. Compute/visualize statistics from results

Dice Fighters Exercise

Red Team:

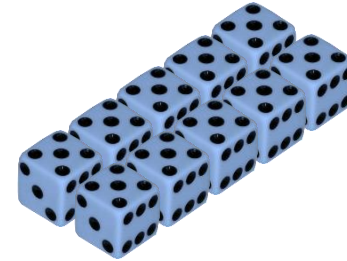
- 2x fighting force size
- Simple weapons



- Roll 6 to hit target

Blue Team:

- Small fighting force
- 3x effective weapons



- Roll 4|5|6 to hit target

Q: What is the probability of Red winning?

Modeling Dice Fighters



Elementary state variables

R_t : red team size at time t

B_t : blue team size at time t

Random variables

$h_{R,t}$: number of red hits at time t

$$h_{R,t} \sim \text{binomial} \left(p = \frac{1}{6}, n = R_t \right)$$

$h_{B,t}$: number of blue hits at time t

$$h_{B,t} \sim \text{binomial} \left(p = \frac{2}{3}, n = B_t \right)$$

Derived state variables

$$W = \begin{cases} \text{red} & \text{if } R_{final} > 0 \\ \text{blue} & \text{if } B_{final} > 0 \\ \text{tie} & \text{otherwise} \end{cases}$$

$$X = \begin{cases} 1, & W = \text{red} \\ 0, & \text{otherwise} \end{cases}$$

Initial conditions

$$R_0 = 20, \quad B_0 = 10$$

State changes

$$\begin{aligned} R_{t+1} &= R_t - h_{B,t}, \\ B_{t+1} &= B_t - h_{R,t} \end{aligned}$$



Dice Fighters Samples

Sample (i)	...	W	W=red (X)	Mean (\bar{x})	Std. Dev. (s_x)	Std. Error ($s_{\bar{x}}$)	95% CI
1	...	red	1	1.00	-	-	-
2	...	red	1	1.00	0.00	0.00	[1.00, 1.00]
3	...	blue	0	0.67	0.58	0.33	[0.01, 1.32]
4	...	red	1	0.75	0.50	0.25	[0.26, 1.24]
5	...	red	1	0.80	0.48	0.20	[0.41, 1.19]
6	...	blue	0	0.67	0.52	0.21	[0.25, 1.08]
...

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$$

$$s_{\bar{x}} = \frac{s_x}{\sqrt{N}}$$

$$\bar{x} \pm z_{1-\alpha/2} s_{\bar{x}}$$

$$s_x = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$$

Dice Fighters MC (Excel)

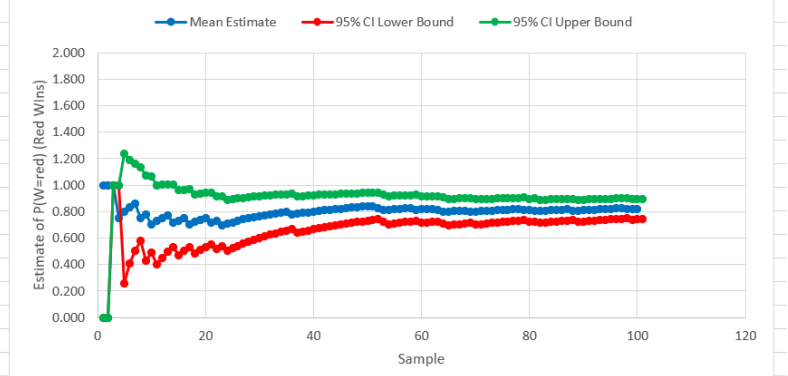


	A	B	C	D	E	F	G	H
1	blue_size	blue_chance_hit	red_size	red_chance_hit	blue_r1	red_r1	blue_r2	red_r2
2	10	0.5	20	0.167	7	17	4	12
3	10	0.5	20	0.167	5	12	3	10
4	10	0.5	20	0.167	8	17	4	14
5	10	0.5	20	0.167	9	15	8	9
6	10	0.5	20	0.167	5	16	0	14
7	10	0.5	20	0.167	7	14	7	10
8	10	0.5	20	0.167	7	16	4	15
9	10	0.5	20	0.167	8	14	6	10
10	10	0.5	20	0.167	6	15	3	11
11	10	0.5	20	0.167	9	16	6	13
12	10	0.5	20	0.167	6	14	4	11
13	10	0.5	20	0.167	5	13	3	11
14	10	0.5	20	0.167	5	17	4	14
15	10	0.5	20	0.167	7	15	5	13
16	10	0.5	20	0.167	7	13	5	10
17	10	0.5	20	0.167	6	18	5	14
18	10	0.5	20	0.167	9	13	6	8
19	10	0.5	20	0.167	9	15	6	11
20	10	0.5	20	0.167	6	16	4	14
21	10	0.5	20	0.167	5	15	3	10
22	10	0.5	20	0.167	8	15	6	12
23	10	0.5	20	0.167	4	12	3	10
24	10	0.5	20	0.167	8	13	7	8
25	10	0.5	20	0.167	8	14	4	11
26	10	0.5	20	0.167	6	16	4	12
27	10	0.5	20	0.167	7	14	4	11

Initial state variables

W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG
blue_r10	red_r10	W		Sample	W=red	Mean Estimate	Standard Deviat	Standard Error	95% CI Lower Bound	95% CI Upper Bound
-8	9	red		1	1	1.000	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
-7	7	red		2	1	1.000	0.000	0.000	1.000	1.000
-13	11	red		3	1	1.000	0.000	0.000	1.000	1.000
4	-8	blue		4	0	0.750	0.500	0.250	0.260	1.240
-22	14	red		5	1	0.800	0.447	0.200	0.408	1.192
-2	2	red		6	1	0.833	0.408	0.167	0.507	1.160
-12	12	red								
4	-13	blue								
-9	8	red								
6	-11	blue								
-13	10	red								
-14	7	red								
-4	11	red								
1	-2	blue								
-7	8	red								
-5	7	red								
2	-6	blue								
-4	5	red								
-18	13	red								
-7	8	red								
2	1	blue								
-9	7	red								
6	-17	blue								
-14	10	red		23	0	0.070	0.470	0.070	0.303	0.600
-14	9	red		24	1	0.708	0.464	0.095	0.523	0.894
-14	9	red		25	1	0.720	0.458	0.092	0.540	0.900
-7	7	red		26	1	0.731	0.452	0.089	0.557	0.905

Derived State



State updates (10 rounds)

Rolling statistics

Dice Fighters MC (Python)



```
def gen_battle():
    global red_size, blue_size

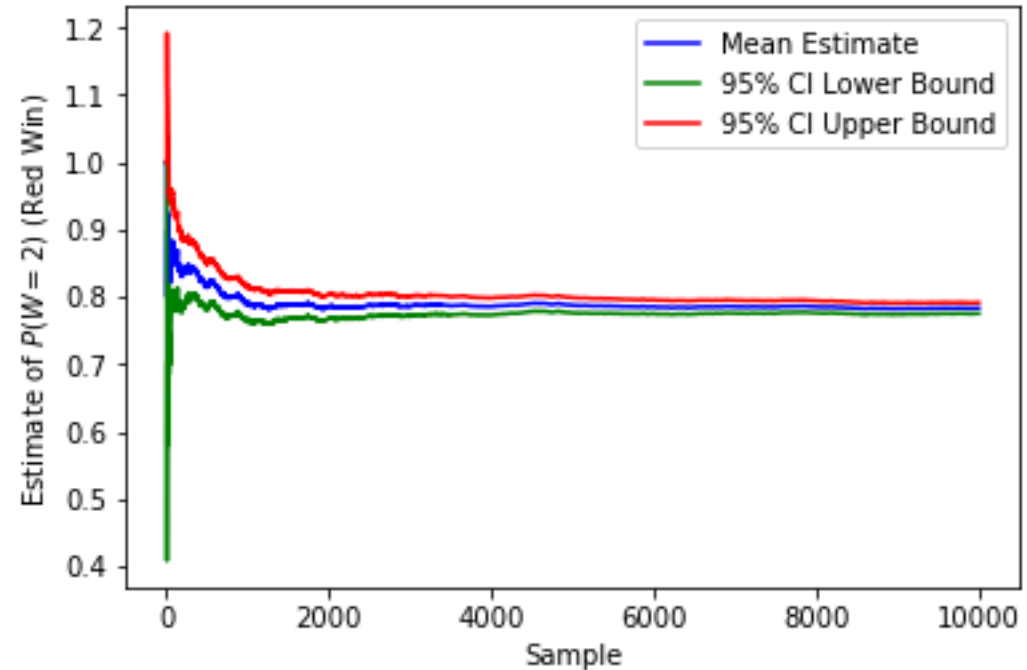
    red_size = 20
    blue_size = 10

    while not is_complete():
        red_hits = gen_red_hits()
        blue_hits = gen_blue_hits()
        red_size -= blue_hits
        blue_size -= red_hits

    if red_size > 0:
        return 'red'
    elif blue_size > 0:
        return 'blue'
    else:
        return 'tie'

samples = np.array([gen_battle()
                    for i in range(10000)])

print(np.mean(samples=='red'))
print(stats.sem(samples=='red'))
```



$$P(W = \text{red}) = 0.783 \pm 0.008 \text{ (95\% CI)}$$

$$P(W = \text{blue}) = 0.213 \pm 0.008 \text{ (95\% CI)}$$

$$P(W = \text{tie}) = 0.004 \pm 0.001 \text{ (95\% CI)}$$



Buffon's Needle Drop



Example: Buffon's Needle

Suppose the floor is made of parallel strips of wood, each the same width T , and we drop a needle of length L onto the floor.

What is the probability that the needle will lie across a line between two strips?

George-Louis Leclerc,
Compte de Buffon, c. 1777

→ Will help us estimate π !

Consider “short needle” cases with $L \leq T$

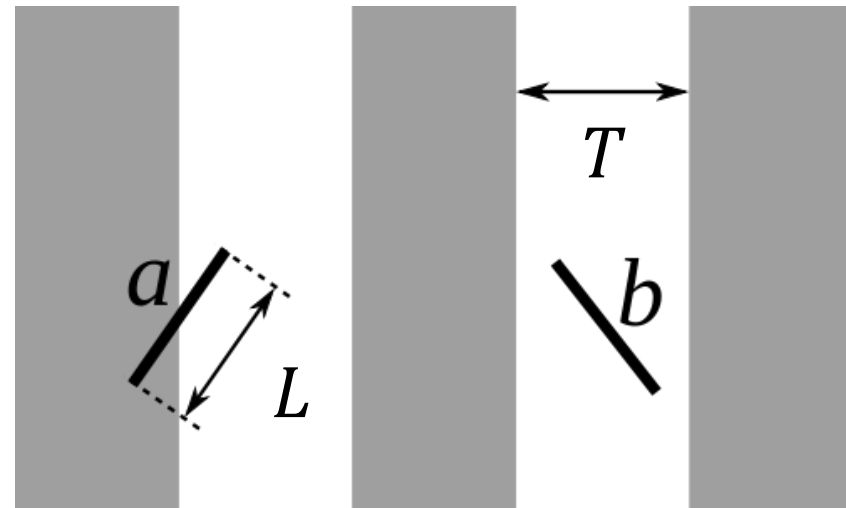
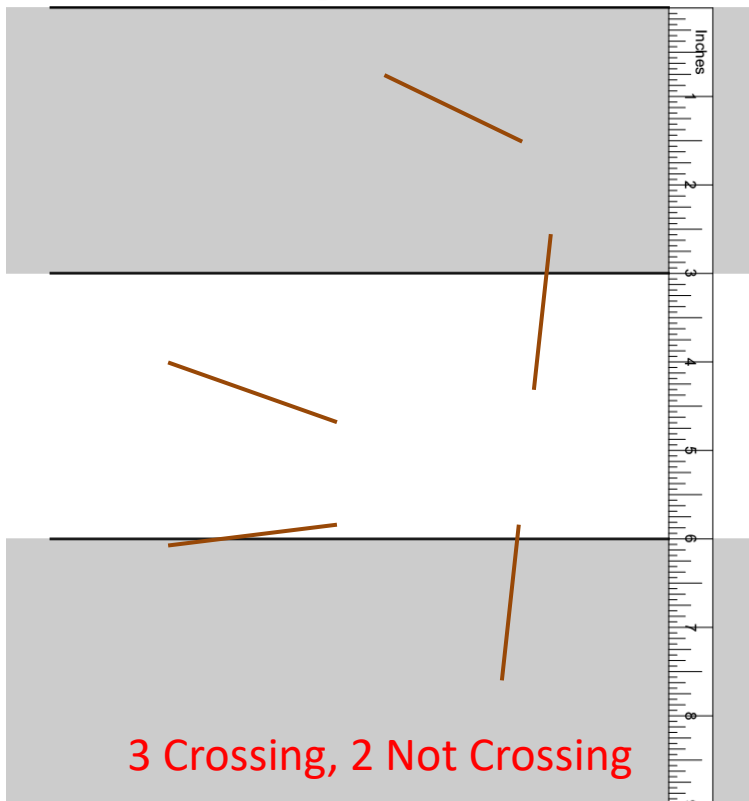


Image: Claudio Rocchini / Wikimedia

Buffon's Needle Experiment



Lines are $T = 3$ in. wide
Needles are $L = 2.5$ in. long



3 Crossing, 2 Not Crossing

Submit at

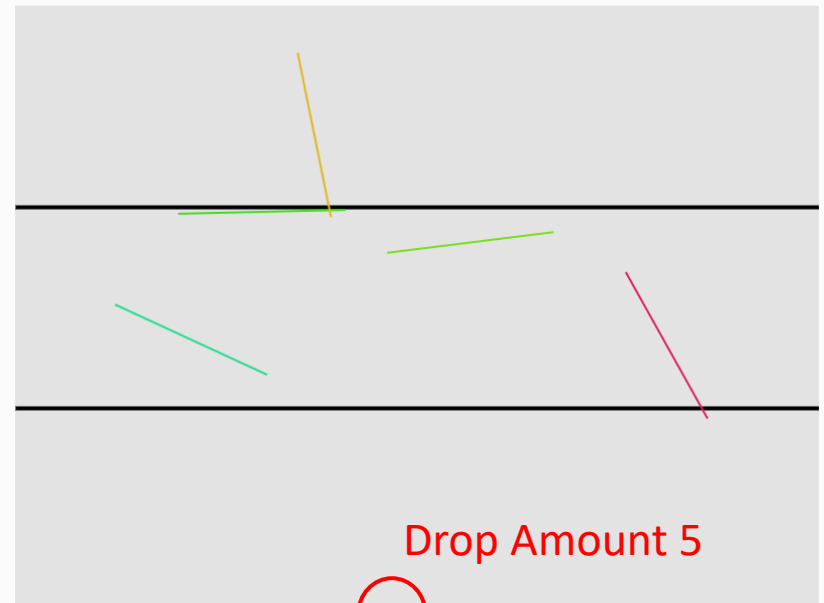
pollev.com/pgrogan

mste.illinois.edu/activity/buffon/

Simulation

In this simulation, press one of the buttons labeled "Drop" to drop a batch of needles on the parallel lines. The measurements and calculations will be completed for you and displayed below the illustration. Each batch of needles you drop will add to the total number of needles measured, allowing you to approximate pi more precisely with each drop. The illustration will show the most recent batch of needles dropped.

Further down, you can also change the scale of the needles dropped or restart the experiment from the beginning. Note that if you change the needle scale, the experiment will automatically reset itself the next time you drop needles, because all the needles need to be the same size and shape for the calculations to work.



Drop Amount 5

Drop 1	Drop 10	Drop 100	Drop 1000	Drop Amount 5	Drop
Measurement					
Needle Scale	0.8333333333333333				
Extent = Perimeter / Greatest Vertex Distance	1				
Number of Drops	5				
Number of Hits	2				
Drops / Hits	2.5				
$\pi \approx 2 * \text{Extent} * \text{Scale} * \text{Drops} / \text{Hits}$	4.166666666666667				
Needle Scale	0.8333333333333333				
Start	New				

Needle Scale 0.833333333

Drop Shape Straight Needles V-Shapes W-Shapes 3-4-5 Triangles Circles Draw Custom Drop Shape