Monte Carlo Simulation

SYS 611: Systems Modeling and Simulation

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Agenda

1. Monte Carlo Simulation
2. Buffon’s Needle Activity

Reference: S.M. Ross, “Statistical Analysis of Simulated Data,” Ch. 8 in *Simulation*, 2012.

J.V. Farr, “Review of Probability and Statistics,” Ch. 3 in Simulation of Complex Systems and Enterprises, Stevens Institute of Technology, 2007.

Monte Carlo Simulation

Monte Carlo Simulation

Monte Carlo simulation solves a problem (possibly deterministic in nature) by statistically analyzing samples from a stochastic model

- Developed in 1940s classified research by von Neumann, Ulam, Fermi, Metropolis (& others)
- Code named by Ulam and Metropolis in reference to *Monte Carlo* casino in Monaco
- Early applications limited due to computation (ENIAC: 1^{st} general-purpose computer in 1946)

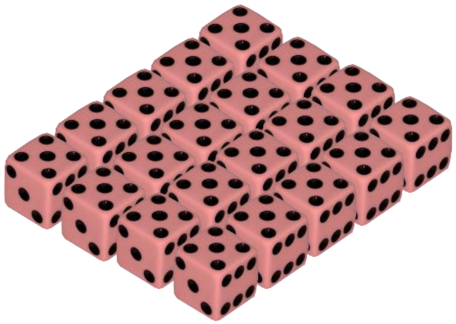
Monte Carlo Approach

1. Identify **elementary state variables** and **random (state) variables** with **probability distributions**
2. Identify **derived (state) variables** and their functional form
3. Determine **number of samples** required or other convergence criteria
4. For each sample, **generate RVs** and compose and **record derived (state) variables**
5. Compute/visualize statistics from results

Dice Fighters Exercise

Red Team:

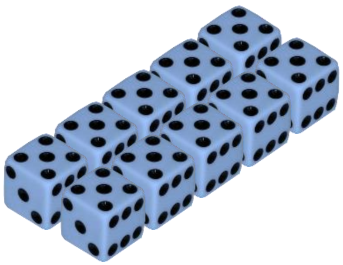
- 2x fighting force size
- Simple weapons



- Roll 6 to hit target

Blue Team:

- Small fighting force
- 3x effective weapons



- Roll 4|5|6 to hit target

Q: What is the probability of Red winning?

Modeling Dice Fighters

Elementary state variables

R_t: red team size at time t

B_t: blue team size at time t

Random variables

$h_{R,t}$: number of red hits at time t

$$h_{R,t} \sim \text{binomial}\left(p = \frac{1}{6}, n = R_t\right)$$

$h_{B,t}$: number of blue hits at time t

$$h_{B,t} \sim \text{binomial}\left(p = \frac{2}{3}, n = B_t\right)$$

Derived state variables

$$W = \begin{cases} \text{red} & \text{if } R_{final} > 0 \\ \text{blue} & \text{if } B_{final} > 0 \\ \text{tie} & \text{otherwise} \end{cases}$$

$$X = \begin{cases} 1, & W = \text{red} \\ 0, & \text{otherwise} \end{cases}$$

Initial conditions

$$R_0 = 20, \qquad B_0 = 10$$

State changes

$$R_{t+1} = R_t - h_{B,t},$$
$$B_{t+1} = B_t - h_{R,t}$$

Dice Fighters Samples

Sample (i)	…	W	W=red (X)	Mean ($\bar{x}$)	Std. Dev. (s_x)	Std. Error ($s_{\bar{x}}$)	95% CI
1	…	red	1	1.00	-	-	-
2	…	red	1	1.00	0.00	0.00	[1.00, 1.00]
3	…	blue	0	0.67	0.58	0.33	[0.01, 1.32]
4	…	red	1	0.75	0.50	0.25	[0.26, 1.24]
5	…	red	1	0.80	0.48	0.20	[0.41, 1.19]
6	…	blue	0	0.67	0.52	0.21	[0.25, 1.08]
…	…	…	…	…	…	…	…

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$s_{\bar{x}} = \frac{s_x}{\sqrt{N}}$$

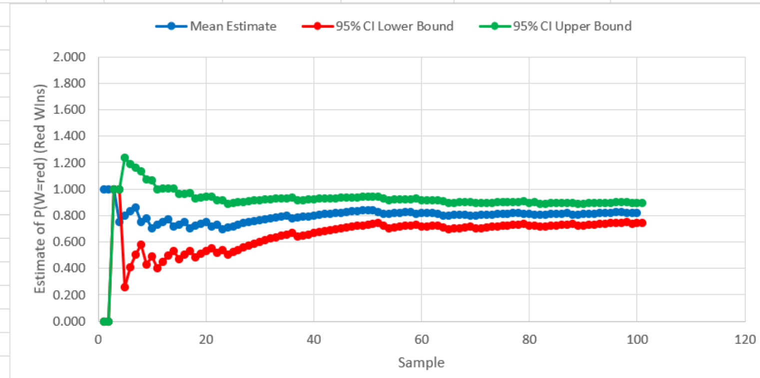
$$\bar{x} \pm z_{1-\alpha/2} s_{\bar{x}}$$

$$s_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

Dice Fighters MC (Excel)

	A	B	C	D	E	F	G	H
1	blue_size	blue_chance_hit	red_size	red_chance_hit	blue_r1	red_r1	blue_r2	red_r2
2	10	0.5	20	0.167	7	17	4	12
3	10	0.5	20	0.167	5	12	3	10
4	10	0.5	20	0.167	8	17	4	14
5	10	0.5	20	0.167	9	15	8	9
6	10	0.5	20	0.167	5	16	0	14
7	10	0.5	20	0.167	7	14	7	10
8	10	0.5	20	0.167	7	16	4	15
9	10	0.5	20	0.167	8	14	6	10
10	10	0.5	20	0.167	6	15	3	11
11	10	0.5	20	0.167	9	16	6	13
12	10	0.5	20	0.167	6	14	4	11
13	10	0.5	20	0.167	5	13	3	11
14	10	0.5	20	0.167	5	17	4	14
15	10	0.5	20	0.167	7	15	5	13
16	10	0.5	20	0.167	7	13	5	10
17	10	0.5	20	0.167	6	18	5	14
18	10	0.5	20	0.167	9	13	6	8
19	10	0.5	20	0.167	9	15	6	11
20	10	0.5	20	0.167	6	16	4	14
21	10	0.5	20	0.167	5	15	3	10
22	10	0.5	20	0.167	8	15	6	12
23	10	0.5	20	0.167	4	12	3	10
24	10	0.5	20	0.167	8	13	7	8
25	10	0.5	20	0.167	8	14	4	11
26	10	0.5	20	0.167	6	16	4	12
27	10	0.5	20	0.167	7	14	4	11

W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG
blue_r10	red_r10	W		Sample	W=red	Mean Estimate	Standard Deviat	Standard Error	95% CI Lower Bound	95% CI Upper Bound
-8	9	red		1	1	1.000	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
-7	7	red		2	1	1.000	0.000	0.000	1.000	1.000
-13	11	red		3	1	1.000	0.000	0.000	1.000	1.000
4	-8	blue		4	0	0.750	0.500	0.250	0.260	1.240
-22	14	red		5	1	0.800	0.447	0.200	0.408	1.192
-2	2	red		6	1	0.833	0.408	0.167	0.507	1.160
-12	12	red								
4	-13	blue								
-9	8	red								
6	-11	blue								
-13	10	red								
-14	7	red								
-4	11	red								
1	-2	blue								
-7	8	red								
-5	7	red								
2	-6	blue								
-4	5	red								
-18	13	red								
-7	8	red								
2	1	blue								
-9	7	red								
6	-17	blue		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
-14	10	red		24	1	0.708	0.464	0.095	0.523	0.894
-14	9	red		25	1	0.720	0.458	0.092	0.540	0.900
-7	7	red		26	1	0.731	0.452	0.089	0.557	0.905



Initial state variables

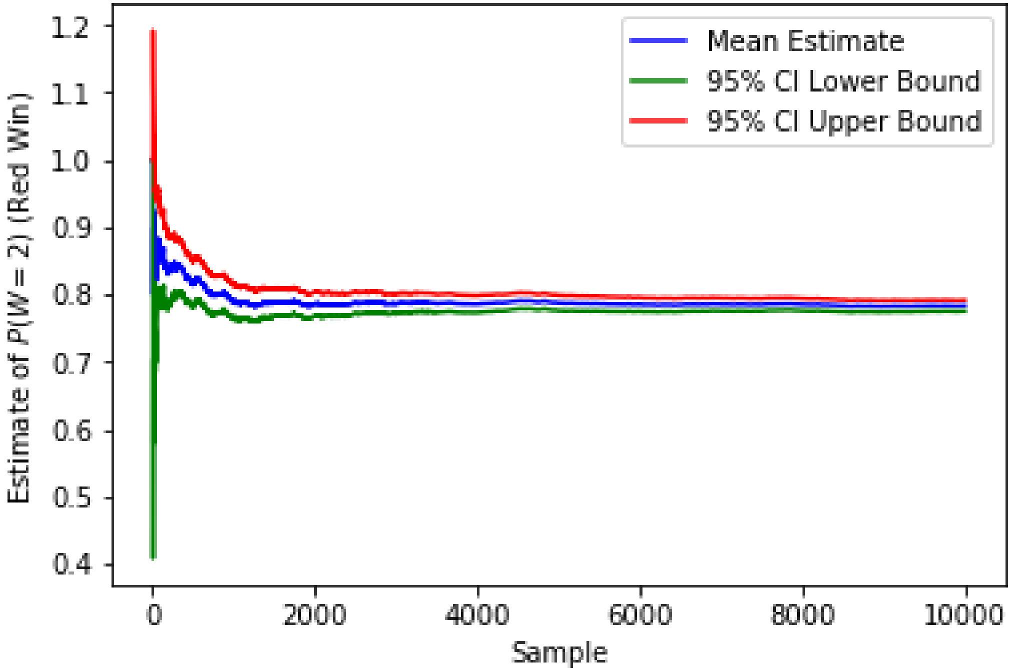
Derived State

State updates (10 rounds)

Rolling statistics

Dice Fighters MC (Python)

```
def gen_battle():
  global red_size, blue_size

  red_size = 20
  blue_size = 10

  while not is_complete():
    red_hits = gen_red_hits()
    blue_hits = gen_blue_hits()
    red_size -= blue_hits
    blue_size -= red_hits

  if red_size > 0:
    return 'red'
  elif blue_size > 0:
    return 'blue'
  else:
    return 'tie'

samples = np.array([gen_battle()
        for i in range(10000)])

print(np.mean(samples=='red'))
print(stats.sem(samples=='red'))
```



$$P(W = \text{red}) = 0.783 \pm 0.008 \text{ (95\% CI)}$$

$$P(W = \text{blue}) = 0.213 \pm 0.008 \text{ (95\% CI)}$$

$$P(W = \text{tie}) = 0.004 \pm 0.001 \text{ (95\% CI)}$$

Buffon's Needle Drop

Example: Buffon's Needle

Suppose the floor is made of parallel strips of wood, each the same width T, and we drop a needle of length L onto the floor.

What is the probability that the needle will lie across a line between two strips?

George-Louis Leclerc, Compte de Buffon, c. 1777

→ Will help us estimate π!

Consider "short needle" cases with $L \leq T$

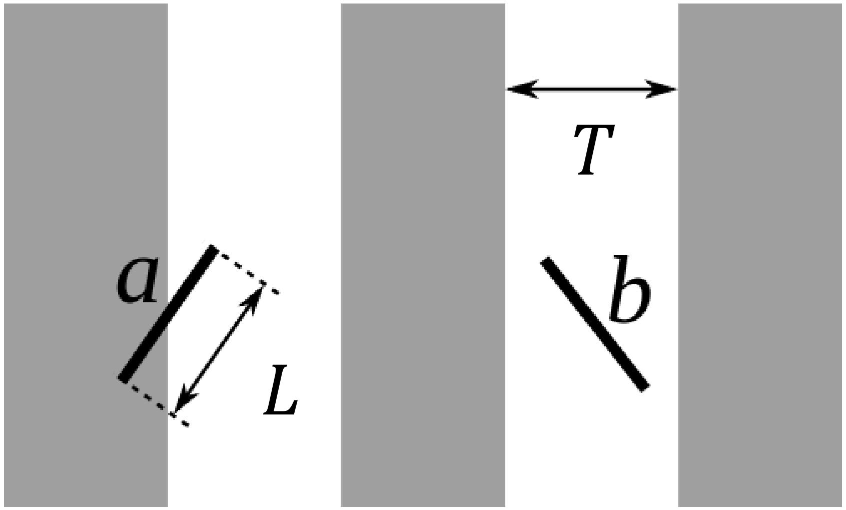
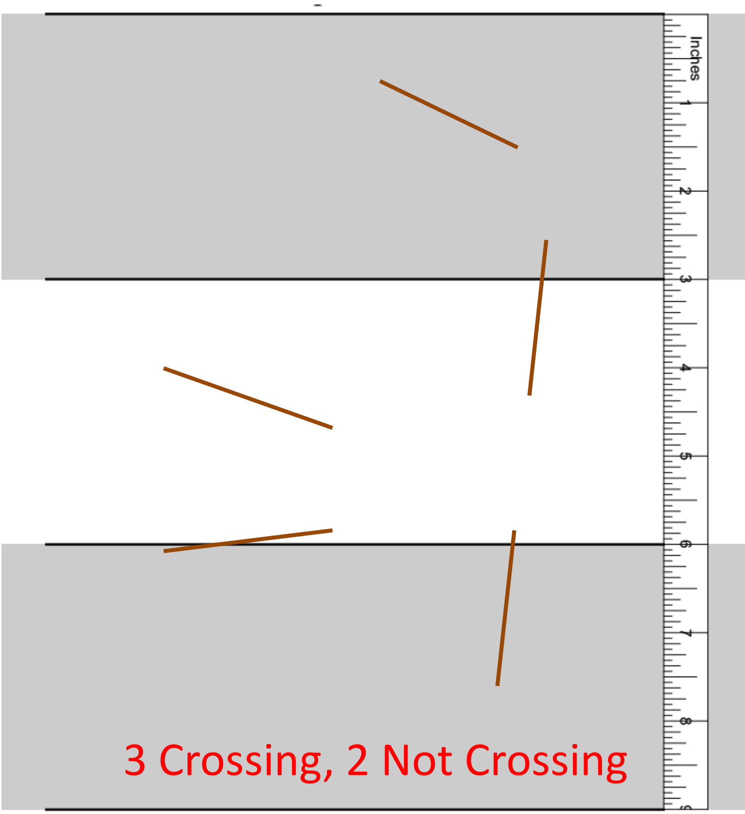


Image: Claudio Rocchini / Wikimedia

Buffon's Needle Experiment

Lines are $T = 3$ in. wide
Needles are $L = 2.5$ in. long



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pollev.com/pgrogan

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